Final Exam of Information Theory in 2023 Fall

Xuan Guang

Collating: Mathzwj

1. Let $p(x,y)$ be given by

$$\frac{1}{8}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(1) Calculate $H(X), H(Y), H(X|Y)$ and $I(X;Y)$.

(2) Calculate $D(p_X \| p_Y)$ and $D(p_Y \| p_X)$.

(3) Draw a Venn diagram for quantities in (1).

2. Denote a probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and let $p = \max_{1 \le i \le n} p_i$. Show that:

(1) $H(\mathbf{p}) \ge -p\log p - (1-p)\log(1-p)$.

(2) $H(\mathbf{p}) \ge -\log p$.

(3) $H(\mathbf{p}) \ge 2(1-p)$.

3. Consider a random variable X that takes six values $\{A, B, C, D, E, F\}$ with probabilities $0.3, 0.25, 0.2, 0.1, 0.1, 0.05$.

(1) Construct a binary Huffman code for the random variable and find its average length.

(2) Construct a quaternary Huffman code for the random variable [i.e., a code over an alphabet of four symbols (call them a, b, c and d)] and find its average length.

(3) Construct a binary code for the random variable by starting with the quaternary Huffman code in (2) and converting the symbols into binary using the mapping

$a \to 00, b \to 01, c \to 10$ and $d \to 11$. Find the average length of the binary code constructed by this process.

(4) For any random variable Y, let L_H be the average length of the binary Huffman code for Y, and let L_{QB} be the average length of the binary code constructed by first building a quaternary Huffman code and converting it to binary using the mapping in (3). Show that $L_H \le L_{QB} \le L_H + 1$.

4. (Han's Inequality) For a subset α of $\mathcal{N}_n = \{1, 2, \ldots, n\}$, denote $(X_i, i \in \alpha)$ by X_α. For $1 \le k \le n$, let

$$H_k = \binom{n}{k}^{-1} \sum_{\alpha:|\alpha|=k} \frac{H(X_\alpha)}{k}.$$

Prove that $H_1 \ge H_2 \ge \ldots \ge H_n$.

Note: In the exam, you only need to handle the case $n = 4$.

5. A code is a fix-free code if it is both a prefix code and a suffix code. Let $l_1, l_2, \ldots,$ l_m be m positive integers. Prove that if

$$\sum_{k=1}^{m} 2^{-l_k} \le \frac{1}{2},$$

then there exists a binary fix-free code with codeword lengths $l_1, l_2, \ldots, l_m$.

6. (1) A channel has the following probability transition matrix:

$$\frac{1}{8}\begin{bmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \end{bmatrix}$$

Calculate its capacity.

(2) A channel with output alphabet $\mathcal{Y}$ and probability transition matrix $p(y|x)$ is said to be weakly symmetry of type II if there is a division $\mathcal{Y} = \bigcup_{i=1}^{n} \mathcal{Y}_i$ such that for $1 \le i \le n$, the part of $p(y|x)$ corresponding to $\mathcal{Y}_i$ satisfies that the rows are

permutations of each other and the columns are permutations of each other. Derive a formula for the capacity of a weakly symmetry of type II channel.

7. The *Z*-channel has binary input and output alphabets and the following probability transition matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \quad x, y \in \{0,1\}$$

Find the capacity of the *Z*-channel and the maximizing input probability distribution.