# 2023-2024 交换代数期末考试

1. Let $R = \mathbb{Z}, S = \{2^n, n \in \mathbb{N}\}, T = \{2n+1, n \in \mathbb{Z}\}$

   (a) compute $S^{-1}\mathbb{Z}$ and $T^{-1}\mathbb{Z}$ and decide whether there are local ring.

   (b) write down (or discribe ) $Spec(\mathbb{Z})$ and $Spec(T^{-1}\mathbb{Z})$

   (c) if $X \subseteq \mathbb{C}[x, y]$, then there exists a finite set $I \subseteq \mathbb{C}[x, y]$ such that the zero set $V(X) = V(I)$

2. Let $R$ be a Noetherian ring, prove that

   (a) the formal power series ring $R[[x]]$ is also Noetherian

   (b) suppose $M$ is a finitely generated $R$-module, then $M$ is Noetherian

3. Let $X = (a_{ij})_{2\times 2}$ be a $2 \times 2$ matrix with indeterminates $a_{ij}$, $k$ be a field, $R = k[a_{11}, ..., a_{22}]$ and $I$ be an ideal of $R$ generated by entries of $X^2$

   (a) compute the zero set $V(I)$ of $I$

   (b) prove that $\sqrt{I} = (det(X), Tr(X))$

4. Let $k$ be a field, $R = k[x_1, x_2, x_3, x_4]$ be a polynomial ring

   (a) Compute the hilbert polynomial of the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$

   (b) Let $S = \mathbb{C}[a^3, a^2b, b^3, ab^2]$, prove that $S$ is a $R$-module and write down the free resolution and graded resolution of $S$

   (c) compute the hilbert polynomial of $S$ and the Krull dimension $dim(S)$ of $S$

5. Let $K$ be a field, $X$ a subset of $k^n$

   (a) write down a topology basis of $k^n$ with respect to Zariski topology

   (b) Let $\overline{X}$ be the Zariski closure of $X$, prove that there exist a finite set $\{f_i, 1 \leq i \leq m\} \subseteq k[X_1, ..., X_n]$ such that $\overline{X} = \bigcap_{i=1}^m V(f_i)$

   (c) give an equivalent relation on $GL_n(\mathbb{C})$ such that the Jordan canonical form of a given matrix $A$ is equivalent to the diagonal matrix with diagonal elements the eigenvalue of $A$

6. Let $R/S$ be an integral extension of rings

   (a) prove the incomparability theorem

   (b) Suppose $R$ is a Noetherian domain, prove that $R$ is UFD iff every prime ideals of $R$ of height 1 is principle

   (c) Let $k$ be a field, prove that $dim(k[x_1, ..., x_n]) = n$

7. Let $R$ be a Dedekind domain

(a) prove that $R$ is a Noetherian domain and write down a equivalent criteria for $R$ to be a Dedekind domain

(b) prove that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain but not a UFD

(c) show that $\mathbb{Z}[\sqrt{-5}]$ has class number 2 and the equation $m^3 = n^2 + 5$ has no integers solution

8. Let $R$ be a ring, $M, N, S$ are $R$-modules

(a) give a counterexample showing that $S \otimes_R Hom_R(M, N)$ is not necessarily isomorphic to $Hom_R(S \otimes_R M, S \otimes_R N)$

(b) prove that if $S$ is flat and $M$ is finitely presented, then $S \otimes_R Hom_R(M, N) \simeq Hom_R(S \otimes_R M, S \otimes_R N)$

(c) prove that if $U$ is a multiplicative subset of $R$ and $M$ is finitely presented, then $U^{-1} Hom_R(M, N) \simeq Hom_{U^{-1}R}(U^{-1}M, U^{-1}N)$