Final Examination: Riemann Surface in a Nutshell

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Time: 18:30-21:30

PS: We will keep examination until we are threw out of the second main building by guards. Besides, you can use any language of human as you like. Finally, the examination is in English.

1. $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function with bounded real part. Prove f is a constant.

2. p_1, p_2, p_3 are 3 distinct points on $\mathbb{C}$. Compute the 1st cohomology group $\mathrm{H}^1(X, \mathbb{Z})$, where $X = \mathbb{C} \backslash \{p_1, p_2, p_3\}$.

3. Prove: every compact Riemann surface X with genus 2 admits a 2-sheeted holomorphic covering $f : X \to \mathbb{P}^1$.

4. Consider a tori $\mathbb{C}/\Gamma$. Prove: For any homomorphism $a : \pi_1(\mathbb{C}/\Gamma) \to \mathbb{C}$, there exists a closed 1-form ω such that its periods homomorphism a_ω is equal to a.

5. Consider a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$, are $\ker\phi$, $\mathrm{coker}\phi$ sheaves? Give proof or counter example.

6. Suppose $\mathcal{R}$ is the sheaf of meromorphic functions with residue 0 on a compact Riemann surface X. Conside the homomorphism sequence of sheaves:

$$0 \to \mathbb{C} \hookrightarrow \mathcal{M} \xrightarrow{\mathrm{d}} \mathcal{R} \to 0.$$

(a) Prvoe: the above sequence is exact.

(b) Prove: $\mathrm{H}^1(X, \mathbb{C}) \cong \mathcal{M}(X)/\,\mathrm{d}\mathcal{R}(X)$.

7. Consider sphere $\mathbb{P}^1$ and tori $\mathbb{C}/\Gamma$.

(a) Give an example of non-constant holomorphic map $f : \mathbb{C}/\Gamma \to \mathbb{P}^1$ (The proof is not needed).

(b) Prove: there is no non-constant holomorphic map $g : \mathbb{P}^1 \to \mathbb{C}$.

8. Consider a line bundle L on a compact Riemann surface.

(a) Prvoe: if $\deg(L) < 0$, there exists no global holomorphic section of L.

(b) Prove: if $\deg(L) = 0$, then L is trivial or adimts no non-constant global holomorphic section.

9. Suppose X is a compact Riemann surface with genus $g > 2$. Conside a point $p \in X$.

(a) Prvoe: there are g Wierstass gaps $1 \leq n_1 < n_2 < \cdots < n_g \leq 2g - 1$ of p.

(b) Prove: if $n_2 > 2$, then $n_i = 2i - 1,\ 1 \leq i \leq g$.

10. On a compact Riemann surface X, divisor D is called special if there exists a holomorphic 1-form $\omega \in \Omega(X)$, such that $(\omega) \geq D$.

(a) Prvoe: if $\deg(D) < g$, D is special.

(b) Prove: special divisor D satisfies $\deg(D) \leq 2g - 2$.

11. Suppose X is a compact Riemann surface with genus g. $p_1, \ldots, p_n$ are n distinct points on X and $c_1, \ldots, c_n$ are n points on $\mathbb{P}^1$. Prove there exists a holomorphic map $f : X \to \mathbb{P}^1$ such that $f(p_i) = c_i$.
(Hint: first construct a holomorphic map $f : X \to \mathbb{P}^1$ with $f(p_1) = 1$ and $f(p_i) = 0,\ i > 1$.)